

Scaling in a toy model of gluodynamics at finite temperatures

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Abstract

In the limit of $\xi \simeq a_\sigma/a_\tau \rightarrow \infty$ the gluodynamics without the magnetic part of action ($S_M \sim 1/\xi$) is considered as a self-contained model. The model is studied analytically in the continuum limit on an extremely large lattice ($N_\tau \rightarrow \infty$). Scaling conditions for critical temperature and string tension are considered. The model shows trivial ($g^2 \sim a_\tau$) asymptotic freedom in the case of continuous gauge groups and nontrivial one ($g^2 \sim 1/\ln 1/a_\tau$) for discrete groups.

1 Introduction. MC experiment and some problems of renormalisation procedure in LGT

Since the confirmation of the approach to the continuum limit is of particular importance in LGT, the systematic scaling analysis at larger $\beta = \frac{2N}{g^2}$ seems inevitable. In lattice QCD the non-perturbative aspects of the theory are of primary interest, but the renormalisation-group techniques are added, as a rule, in a perturbative way. Feasible enough description of the running coupling behavior by field theory methods allows to establish a direct connection with the standard formalism of perturbation theory and the renormalisation procedure in lattice gauge theory (LGT). Now the Callan-Symanzik beta function

$$\beta_{CS}(g) = -a \frac{\partial g}{\partial a} = -b_0 g^3 - b_1 g^5 - b_2 g^7 - \dots \quad (1)$$

which leads to

$$g^{-2} = b_0 \ln \frac{1}{a\Lambda_L} + \frac{b_1}{b_0} \ln \ln \frac{1}{a\Lambda_L} + \dots \quad (2)$$

with

$$b_0 = \frac{11}{3} \frac{N}{16\pi^2}; \quad b_1 = \frac{34}{3} \left(\frac{N}{16\pi^2} \right)^2 \quad (3)$$

is computed in a perturbative theory with an accuracy even greater than that required in LGT at present. Outside the weak coupling region, however, the scaling behavior is described by the beta function, which, generally speaking, is unreachable for the perturbation theory.

In the lattice gauge theory, the conditions, which provide asymptotic scaling in a strong coupling regime were established long ago (see, e.g., [1]). In the case of anisotropic lattice $a_\sigma / a_\tau \neq 1$ such conditions can be written as

$$\beta_{\sigma,\tau} \simeq \bar{\beta}_{\sigma,\tau} \exp(a_\sigma a_{\sigma,\tau} M_{\sigma,\tau}^2) \rightarrow \bar{\beta}_{\sigma,\tau} = const; \quad M_{\sigma,\tau} = const \quad (4)$$

$$\beta_{\sigma,\tau} \simeq \bar{\beta}_{\sigma,\tau} \exp(a_{\sigma,\tau} m_{\sigma,\tau}) \rightarrow \bar{\beta}'_{\sigma,\tau} = const; \quad m_{\sigma,\tau} = const \quad (5)$$

Although (5) and (4) guarantee the scaling behavior of mass gaps $m_{\sigma,\tau}$ and string tensions $\alpha_{\sigma,\tau}$ they cannot be satisfied simultaneously. In both cases the Callan-Symanzik beta function $\beta_{CS} \simeq \beta_{\sigma,\tau} \ln \frac{\bar{\beta}_{\sigma,\tau}}{\beta_{\sigma,\tau}}$ turns to zero at the finite value of bare coupling, which scarcely can be regarded as a 'strong' one. Moreover, since some physical quantities have different strong coupling expansions, it is hardly possible to adjust the couplings in such a way, that

all of them will remain simultaneously invariant under changing ξ [2]. All this makes us doubt that the continuum limit may be realized in the strong coupling area.

Even being computed in a weak coupling area Eq. (1) may differ in different approaches. In particular, only two leading terms in (1) are universal. Such disturbances are annoying but hardly significant, because the goal of lattice calculation is to discover the value of some physical observable when the UV cutoff is taken very large, so that the physics is, indeed, in the first term of (1). Everything else is an artifact of calculation.

However, although the continuum limit ($a \rightarrow 0$) in asymptotically free theories corresponds to $g \rightarrow 0$ such theories in the continuum limit do not become perturbative [3]. For example, while calculating $R \times R$ Wilson loop expectation value in perturbation theory, the correction to the leading term typically has the form

$$g^2 \ln R = g^2 \ln \frac{r}{a} \rightarrow \infty; \quad r \rightarrow \infty, \quad (6)$$

so that this correction becomes arbitrary large at large distances.

At short distances, i.e. for energy scales larger than the heavy-quark mass, the physics is perturbative and described by ordinary QCD. For mass scales much below the heavy-quark mass, the physics is essentially non-perturbative because of confinement. Thus, although a trend in approaching to the asymptotic scaling in 2-loop approximation is seen at $g^2 \sim 1$, one cannot exclude that the non-perturbative effects may seriously change Eq. (1) at extremely small g^2 .

At the moment, lattice simulations are the most popular way of extracting truly non-perturbative results from quantum field theories. The lattice calculation of a number of physical values, e.g. of the glueball mass, string tension, etc., at different g may be used to find the beta function, these quantities, however, reflect the long distance properties. The exception is the Creutz ratio of Wilson loops in which the self mass contributions are cancelled, so it is, indeed, a candidate for the study of beta function. Although many interesting attempts are made to obtain the QCD running coupling from the first principles by lattice computations, however, two old problems [4] are still far from an ultimate solution: the ratios of small loops are contaminated by lattice artifacts which results in the systematic error $\sim \frac{2N}{g^2}$ and finite size effects (if the correlation length is comparable with the lattice size).

Detailed analysis [5] (see also [6]) of the recent developments in renormalisation technique shows that an important uncertainty arises due to the fact

that the results of renormalisation procedure depend on the details of the lattice regularization. Since so called *bare perturbation theory* is regarded as unreliable, various recipes, based on mean-field theory or resummations of tadpole diagrams, have been suggested to deal with this problem [7, 8]. However, different prescriptions give different results and it is in any case unclear how errors can be reliably assessed.

An interesting method to compute renormalization factors that does not rely on bare perturbation theory has been proposed in [9]. However this method has its own problems, the most important being that the momentum p should be significantly smaller than $1/a$ to suppress the lattice effects, but not too small since in this case you cannot apply the renormalized perturbation theory with desired confidence.

A method which may provide general solution of the non-perturbative renormalization problem through a recursive procedure [10] is suggested in [11, 12, 13, 14]. In such a treatment the scale evolution of the renormalized parameters and operators can be studied by changing the lattice size L at fixed bare parameters. With an accuracy up to lattice effects the running couplings on the two lattices are then related through

$$g^2(2L) = \sigma(g^2(L)), \quad (7)$$

where σ is an integrated form of the Callan-Symanzik beta function. The recursive procedure has been used to compute the running coupling in quenched QCD [11, 12] and the running quark masses [13, 14]. An important point to note is that the renormalization group invariant quark masses M are scheme-independent. Such procedure, however, is not totally independent of the initial choice of the scaling functions $\sigma(g^2(L))$ in (7).

Let us consider some problems of renormalisation in LGT immediately revealed by the MC experiment. Comprehensive studies LGT on lattices of up to $48^3 \times 16$ at finite temperature [15] and up to $48^3 \times 56$ at zero temperature [16] show that (asymptotic) terms violating the scaling are still strong at β values accessible for todays numerical simulations. These terms manifest themselves most strikingly in the famous dip of the “step function”, $\Delta_\beta \equiv \beta_{CS}(a) - \beta_{CS}(2a)$ referring to the deconfinement temperature [15]. This pattern is similar in all $SU(N)$ gauge theories. The strongest sensitivity was observed in the $N_\tau = 4$ case, where the numerical calculations close to T_c are performed with bare couplings $\beta \sim 5.6 \div 5.8$, i.e. in the region of the strongest variation of the cut-off with β (dip in the Δ_β) [15]. Besides, with the ‘dip’ in Δ_β the presence of even a zero in Δ_β within this region of β may be suspected [17]. Numerical studies showed deviations from (1) on

entering the scaling region where the correlation length ς begins to grow. It is especially worth to note, that these deviations are always of such a pattern that ς increases faster with β than is predicted by (1), as if the theory approaches the critical point and consequently is not asymptotically free [17].

Recent analyses [18] showed that scaling violations persist in physical quantities such as string tension and hadron masses up to $\beta = 6.8$ [19]. Above the deconfining transition point, the matching procedure inevitably suffers large errors. MCRG has been performed by several groups on 16^4 lattices in the large β region up to 7.2 [4, 20, 21, 22, 23]. However, these results were inconclusive and even controversial. Bowler et al. [20, 21] found sizeable deviations from 2-loop scaling. Recently Hoek has reanalyzed the same data and claimed a very slow approach to scaling [23]. Studies on larger lattices with better statistics are required to clarify the scaling behavior in the high region of β .

Unfortunately, with present computer capacity we are still far enough from the scaling region and a -dependent terms are large, so we can hardly finally decide if the perturbative beta function (1) is appropriate for LGT at long distance area. However, if for the calculation of some physical value (e.g. critical temperature T_c) one applies the beta function computed within LGT for another physical value (e.g. string tension σ) essentially better results may be expected. Moreover, σ itself may be used as an instrument of phenomenological renormalisation procedure. Indeed, scaling behavior for $\sqrt{\sigma}/T_c$ is excellent [2], [24]. MC computations in 3 dimensions of ratio $\sqrt{\sigma_3}/T_c$ for the $SU(2)$ and $SU(3)$ cases [24], [25] as a matter of fact, show only minor deviations from scaling. This, however, can be taken as an evidence in favor of the universality of the renormalisation procedure, rather than of the applicability of perturbative beta function.

Unfortunately, on the basis of today's numerical computations, it is difficult to anticipate beta function behavior in the limit of $a \rightarrow 0$, taking perturbative calculations as a guide¹. Therefore in the near future, at least at this point, analytical methods will be still needed. Though one may well believe that deviations from scaling will disappear completely with increasing lattice sizes, a direct check of this suggestion is rather difficult because of limited computer capacity. In our previous article [27] we made an attempt to compute analytically spatial string tension $\sqrt{\sigma}$ in $Z(2)$ and $Z(3)$ gluodynamics at finite temperature in spherical model approximation. As

¹Warnings concerning the enhancement of nonperturbative effects as $a \rightarrow 0$ in the presence of power divergences have been issued [26]

it was shown, the spatial string tension demonstrated scaling behavior for certain parameter area $b_0 \approx \frac{\xi}{2N} \left(1 - \cos \frac{2\pi}{N}\right)^{-1}$. The numeric value of b_0 with anisotropy parameter $\xi \simeq 1$ agree with the standard value $b_0 = \frac{11}{12} \frac{N}{4\pi^2}$. In this article we try to find analytically the running coupling dependence on lattice sizes taking some simple cases and compare such a dependence for models with continuous and discrete gauge groups. For this purpose we reconsider the well known effective action for finite temperature gluodynamics [28], that helps us to estimate the critical temperature and running coupling behavior in continuum limit on 'large' lattices ($N_\tau \rightarrow \infty$).

The most serious problem appearing in such approach is the calculation of the contribution of the magnetic part (S_M) of the action. It was studied in [30] where it had been established that perturbative corrections slowed down the critical temperature dependence on N_τ . However, there are reasons to doubt that the magnetic part contribution can be reliably computed perturbatively. Below we give some evidence in favor of the suggestion that in $N_\tau \rightarrow \infty$ limit the situation can hardly be totally cured by the computation of the finite set of perturbative series terms.

Being unable to make such calculations nonperturbatively we consider an 'electric toy' model, where all terms of order $1/\xi^2$ in QCD action are omitted. In particular the magnetic part of the action in this model is absent. Although such approach can be regarded as too rough and unrealistic, it may be interesting from mathematical point of view, as an example of a model, where renormalisation procedure can be performed analytically entirely within the lattice theory. In fact, very similar approximation was partially used for the analysis of pure gauge theories in [42] where $T = \frac{1}{a_\tau N_\tau} \rightarrow \infty$ was additionally claimed. As it was mentioned above, along with $a_\tau \rightarrow 0$ we consider the limit of $N_\tau \rightarrow \infty$, therefore in suggested model the finite temperatures area also becomes accessible. Some results obtained in such toy model may be useful for general considerations of renormalisation procedure in LGT.

2 Effective action for finite temperature $Z(N)$ gluodynamics

Here we neglect the contribution of space-like plaquettes i.e. the chromomagnetic part of the action. Of course, one may do it in a strong coupling

²Asymmetry (or anisotropy) parameter is defined as $\xi \simeq a_\sigma/a_\tau$, where a_τ and a_σ are temporal and spatial lattice spacings.

approximation, however, as it was repeatedly pointed out, this regime is dangerous to work in and as it has been done in [42] (see also [30] and [31]), we use the anisotropic lattice

$$\beta_\tau \equiv \beta \equiv \frac{2N\xi}{g^2} \gg \beta_\sigma \equiv \frac{2N}{g^2\xi} \quad (8)$$

where $\xi = \sqrt{\frac{\beta_\tau}{\beta_\sigma}}$ and $g^2 = g_\tau g_\sigma$. In the weak coupling region $g_\tau^2 \approx g_\sigma^2 \approx g^2 + O(g^4)$ [32], [33] so $\xi \approx \frac{a_\sigma}{a_\tau}$.

As it is well known, the action for $Z(N)$ gluodynamics in such an approximation may be written as

$$-S \approx -S_E = \beta \text{Re} \sum_{\tau=0}^{N_\tau-1} \sum_{\mathbf{x}, \tau, n} z_0(\mathbf{x}, \tau) z_n(\mathbf{x}, \tau+1) z_0(\mathbf{x}+n, \tau)^* z_n(\mathbf{x}, \tau)^* \quad (9)$$

and having imposed Hamiltonian gauge condition

$$z_0(\mathbf{x}, \tau) = 1 - \delta_\tau^0 + \delta_\tau^0 \Omega(\mathbf{x}); \quad \Omega(\mathbf{x}) = \prod_{\tau=0}^{N_\tau-1} z_0(\mathbf{x}, \tau); \quad \Omega(\mathbf{x}) \in Z(N), \quad (10)$$

we get

$$-S_E = \beta \text{Re} \sum_{\mathbf{x}, n} \{ \Omega(\mathbf{x}) z_n(\mathbf{x}, 0) \Omega(\mathbf{x}+n)^* z_n(\mathbf{x}, N_\tau-1)^* \} - S_{ch} \quad (11)$$

The action (11) is a set of one-dimensional chains

$$-S_{ch} = \beta \text{Re} \sum_{\mathbf{x}, n} \left\{ \sum_{\tau=1}^{N_\tau-1} z_n(\mathbf{x}, \tau) z_n(\mathbf{x}, \tau-1)^* \right\} \quad (12)$$

attached by the first $z_n(\mathbf{x}, 0)$ and last $z_n(\mathbf{x}, N_\tau-1)$ links to the plaquettes placed at $\tau=0$. Beside such links, these plaquettes also contain Polyakov lines $\Omega(\mathbf{x})$. It is evident that from

$$e^{\beta \text{Re} z} \equiv \sum_j \Im_j(\beta) z^j; \quad \sum_{[z]} z^j = \sum_{k=0}^{N-1} \exp\{ij\varphi_k\} = N\delta_0^j; \quad \varphi_k \equiv \frac{2\pi k}{N} \quad (13)$$

follows³

$$\Im_j(\beta) = \frac{1}{N} \sum_{[z]} z^j \exp\{\beta \text{Re} z\} = \frac{1}{N} \sum_{k=0}^{N-1} \exp\{\beta \cos \varphi_k + i\varphi_k j\}, \quad (14)$$

³In the case of, e.g., $Z(3)$ it gives: $\Im_0(\varkappa) = \frac{e^{\varkappa} + 2e^{-\frac{\varkappa}{2}}}{3}$ and $\Im_{\pm 1}(\varkappa) = \frac{e^{\varkappa} - e^{-\frac{\varkappa}{2}}}{3}$

and having for each link

$$\begin{aligned} & \exp \left\{ \tilde{\beta} (1) \operatorname{Re} [z_n (\mathbf{x}, \tau) z_n (\mathbf{x}, \tau - 1)^*] \right\} \\ &= \sum_{j=0}^{N-1} \mathfrak{S}_j (\beta) \cdot z_n (\mathbf{x}, \tau)^j z_n (\mathbf{x}, \tau - 1)^{*j} \end{aligned} \quad (15)$$

the sum over all variables except $z_n (\mathbf{x}, 0)$ and $z_n (\mathbf{x}, N_\tau - 1)$ we get the expression for $(N_\tau - 1)$ -section chain

$$\begin{aligned} & \sum_{[z_n (\mathbf{x}, \tau)]}^I \exp \left\{ \tilde{\beta} (N_\tau - 1) \operatorname{Re} [(N_\tau - 1) z_n (\mathbf{x}, \tau) z_n (\mathbf{x}, \tau - 1)^*] \right\} \\ &= \sum_{j=0}^1 [\mathfrak{S}_j (\beta)]^{N_\tau - 1} z_n (\mathbf{x}, \tau)^j z_n (\mathbf{x}, \tau - 1)^{*j}, \end{aligned} \quad (16)$$

that differs from one-section chain simply by the substitution $\mathfrak{S}_j (\beta) \rightarrow [\mathfrak{S}_j (\beta)]^{N_\tau - 1}$. Summing over the remaining chain variables one easily gets the familiar expression

$$\begin{aligned} \exp \{-S_{eff}\} &= \mathfrak{S}_0 (\beta)^{N_\tau} \sum_j \left(\frac{\mathfrak{S}_j (\beta)}{\mathfrak{S}_0 (\beta)} \right)^{N_\tau} \Omega (\mathbf{x})^j \Omega (\mathbf{x} + n)^{*j} \\ &\approx \mathfrak{S}_0 (\beta)^{N_\tau} \exp \left\{ \left(\frac{\mathfrak{S}_1 (\beta)}{\mathfrak{S}_0 (\beta)} \right)^{N_\tau} \Omega (\mathbf{x}) \Omega (\mathbf{x} + n)^* \right\} \end{aligned} \quad (17)$$

and since only small j survive for $N_\tau \rightarrow \infty$ Eq. (17) leads to

$$-S_{eff} \approx \tilde{\beta} (N_\tau) \sum_{\mathbf{x}, n} \Omega (\mathbf{x}) \Omega (\mathbf{x} + n)^* - N_\tau F_0, \quad (18)$$

where the effective coupling $\tilde{\beta} (N_\tau)$ is completely defined by the one-dimensional chain and related to the bare coupling β by the expression

$$\left(\frac{\mathfrak{S}_1 (\beta)}{\mathfrak{S}_0 (\beta)} \right)^{N_\tau} \approx \frac{\mathfrak{S}_1 (\tilde{\beta} (N_\tau))}{\mathfrak{S}_0 (\tilde{\beta} (N_\tau))} \approx \tilde{\beta} (N_\tau); \quad N_\tau \rightarrow \infty \quad (19)$$

and the 'free' term⁴ is

$$-F_0 \equiv \ln \mathfrak{S}_0 (\beta) \quad (20)$$

⁴Only this term survives if we break the chain at $t = t_0$ with open boundary conditions $z_0 (\mathbf{x}, t_0) \neq z_0 (\mathbf{x}, t_0 + N_\tau)$. It easy to see that the first $N_\tau - 1$ terms of 'high temperature' expansion contribute *only* in F_0 .

In the wide area ($\beta \ll 1$) the functions $\mathfrak{I}_j(\beta) \approx I_j(\beta)$ (inter alia $\mathfrak{I}_j(\beta) \approx \frac{\beta^j}{j!}$ for $\beta \ll 1$) and therefore

$$\tilde{\beta}(N_\tau) \approx \beta^{N_\tau} \quad (21)$$

However, in a very important area of $\beta \gg N$ the $\mathfrak{I}_j(\beta)$ values significantly differ from $I_j(\beta)$ and thus we obtain

$$\begin{aligned} \frac{\mathfrak{I}_1(\beta)}{\mathfrak{I}_0(\beta)} &\approx \frac{1 + 2e^{-\beta(1-\cos \frac{2\pi}{N})} \cos\left(\frac{2\pi}{N}\right)}{1 + 2e^{-\beta(1-\cos \frac{2\pi}{N})}} \\ &\approx \exp\left\{-2\left(1 - \cos\left(\frac{2\pi}{N}\right)e^{-\beta(1-\cos \frac{2\pi}{N})}\right)\right\} \end{aligned} \quad (22)$$

and, accordingly, for $N_\tau \gg 1$

$$\tilde{\beta}(N_\tau) \approx \left(\frac{\mathfrak{I}_1}{\mathfrak{I}_0}\right)^{N_\tau} \approx \exp\left\{-2N_\tau\left(1 - \cos\left(\frac{2\pi}{N}\right)\right)e^{-\beta(1-\cos \frac{2\pi}{N})}\right\}. \quad (23)$$

In particular

$$\tilde{\beta}(N_\tau) \approx \begin{cases} \exp\left\{-4N_\tau e^{-2\beta}\right\} & N = 2 \\ \exp\left\{-3N_\tau e^{-\beta\frac{3}{2}}\right\} & N = 3 \end{cases} \quad (24)$$

so we come to $3d$ $Z(N)$ -spin model with an effective coupling given by (23). Such models undergo phase transition at the point⁵ $\tilde{\beta}(N_\tau) = \tilde{\beta}_c$.

Therefore for fixed N_τ for the critical value of the bare coupling β we get

$$\beta_c \approx \frac{\text{const} + \ln\left(\frac{1}{1-\cos\left(\frac{2\pi}{N}\right)}\right)}{1 - \cos\left(\frac{2\pi}{N}\right)} \quad (25)$$

which is similar to the well-known phenomenological rule [34]

$$\beta_c \approx \frac{\text{const}}{1 - \cos\left(\frac{2\pi}{N}\right)} \quad (26)$$

⁵The critical point for (18) is defined as $\tilde{\beta}(N_\tau) = \tilde{\beta}_c(N)$ where $\tilde{\beta}_c(N)$ are numeric constants. In particular $\tilde{\beta}_c(2) \approx 0.2288$ is the critical coupling of $d3$ Ising model and $\tilde{\beta}_c(3) \approx 0.5501$ $d3$ is for Potts model.

For $N \gg 1$ we cannot cut the series in j at (14), but we may expand $\cos\left(\frac{2\pi}{N}j\right) \approx 1 - \frac{1}{2}\left(\frac{2\pi}{N}j\right)^2$

$$\frac{\mathfrak{I}_j(\beta)}{\mathfrak{I}_0(\beta)} \approx \frac{\sum_{k=0}^{N-1} \exp\left\{-\frac{\beta}{2}\left(\frac{2\pi}{N}\right)^2 k^2 + i\left(\frac{2\pi}{N}\right)kj\right\}}{\sum_{n=0}^{N-1} \exp\left\{-\frac{\beta}{2}\left(\frac{2\pi}{N}\right)^2 n^2\right\}} \quad (27)$$

$$= \frac{\theta_3\left(\frac{\beta}{2}\left(\frac{2\pi}{N}\right)^2; \left(\frac{2\pi}{N}\right)j\right)}{\theta_3\left(\frac{\beta}{2}\left(\frac{2\pi}{N}\right)^2; 0\right)}. \quad (28)$$

Taking into account that for the elliptic function [35]

$$\begin{aligned} \theta_3(\lambda; \varphi) &\equiv \sum_{n=-\infty}^{\infty} \exp\left\{-\lambda n^2 + in\varphi\right\} \\ &= \sqrt{\frac{\pi}{\lambda}} \sum_{m=-\infty}^{\infty} \exp\left\{-\frac{(\varphi + 2\pi m)^2}{4\lambda}\right\}, \end{aligned} \quad (29)$$

one can write with good accuracy [36]

$$\theta_3(\lambda; \varphi) \approx \exp\left\{\tilde{\gamma}(\lambda) \cos \varphi + G(\lambda)\right\}, \quad (30)$$

where $\tilde{\gamma}(\lambda)$ and $G(\lambda)$ are smooth analytical functions of λ

$$\begin{aligned} \tilde{\gamma}(\lambda) &\approx \begin{cases} 2 \exp\{-2\lambda\}; & \lambda \gg 1 \\ \frac{1}{2\lambda}; & \lambda \ll 1 \end{cases}; \\ G(\lambda) &\approx \begin{cases} 0 & \lambda \gg 1 \\ \frac{1}{2} \ln \frac{\pi}{\lambda}; & \lambda \ll 1 \end{cases} \end{aligned} \quad (31)$$

and we get for $\tilde{\beta}(N_\tau)$

$$-\ln \tilde{\beta}(N_\tau) \approx -\frac{N_\tau}{j^2} \ln \frac{\mathfrak{I}_j(\beta)}{\mathfrak{I}_0(\beta)} \approx \frac{N_\tau}{2} \left(\frac{2\pi}{N}\right)^2 \tilde{\gamma}\left[\frac{\beta}{2}\left(\frac{2\pi}{N}\right)^2\right], \quad (32)$$

which finally gives

$$-\ln \tilde{\beta}(N_\tau) \approx N_\tau \left(\frac{2\pi}{N}\right)^2 \exp\left\{-\frac{\beta}{2}\left(\frac{2\pi}{N}\right)^2\right\}; \quad \beta \gg N \quad (33)$$

and

$$-\ln \tilde{\beta}(N_\tau) \approx \frac{N_\tau}{2\beta}; \quad \beta \ll N. \quad (34)$$

If we claim that the critical point $\tilde{\beta}(N_\tau) = \tilde{\beta}_c$ corresponds to the phase transition, the dependence of $\tilde{\beta}(N_\tau)$ on N_τ must be removed at least in the continuum limit by a renormalisation procedure which relates N_τ (or a_τ) to the bare coupling β . The conditions that provide for such an independence are discussed later for some simple cases.

In the previous expressions we could not avoid somewhat annoying 'generalities', but we cherish hope that in what follows those may be regarded as 'justified'. Finally we want to stress the fact that the suggested approach is not limited to small N_τ , however, the coupling $\tilde{\beta}(N_\tau)$ in an effective action decrease with N_τ so we must accordingly increase ξ to guarantee that $\tilde{\beta}(N_\tau) \gg \beta_\sigma$.

3 Effective action for finite temperature $U(1)$, $SU(2)$ and $SU(3)$ gluodynamics

Wilson action for $SU(N)$ gauge group may be written in the static gauge $U_0(\mathbf{x}, \tau) = U_0(\mathbf{x})$ as

$$-S_E = \frac{\beta}{N} \sum_{\tau=0}^{N_\tau-1} \sum_{\mathbf{x}, n} \text{ReSp} \left\{ U_0(\mathbf{x}) U_n(\mathbf{x}, \tau+1) U_0^\dagger(\mathbf{x}+n) U_n^\dagger(\mathbf{x}, \tau) \right\} \quad (35)$$

and after the gauge transformation $U_n(\mathbf{x}, \tau) \rightarrow [U_0(\mathbf{x})]^{-\tau} U_n(\mathbf{x}, \tau) [U_0(\mathbf{x}+n)]^\tau$ we are sure to get ⁶

$$-S_E = \sum_{\mathbf{x}, n} \left(\frac{\beta}{N} \text{ReSp} \left\{ \Omega(\mathbf{x}) U_n(\mathbf{x}, 0) \Omega^\dagger(\mathbf{x}+n) U_n^\dagger(\mathbf{x}, N_\tau-1) \right\} - s_{ch}(\mathbf{x}, \mathbf{n}) \right) \quad (36)$$

which, as well as (18), is given by plaquettes, placed on the last links in the temporal direction and represents $\Omega(\mathbf{x})$ -loop interactions with the one-dimensional chains

$$-s_{ch}(\mathbf{x}, \mathbf{n}) = \frac{\beta}{N} \text{ReSp} \left\{ \sum_{\tau=1}^{N_\tau-1} U_n(\mathbf{x}, \tau) U_n^\dagger(\mathbf{x}, \tau-1) \right\} \quad (37)$$

attached to such plaquettes. Having written

$$\exp \{-s_{ch}(\mathbf{x}, \mathbf{n})\} = \sum_j d_j \Im_j(\beta) \chi_j \left(U_n(\mathbf{x}, \tau) U_n^\dagger(\mathbf{x}, \tau-1) \right), \quad (38)$$

⁶The same result may be obtained by imposing the Hamiltonian gauge condition $U_0(\mathbf{x}, \tau) = 1 - \delta_\tau^0 + \delta_\tau^0 \Omega(\mathbf{x})$ with $\Omega(\mathbf{x}) \equiv U_0(\mathbf{x})^{N_\tau}$, from the beginning.

where

$$\mathfrak{S}_j(\beta) = \int \exp \left\{ \frac{\beta}{N} \text{ReSp} \{U_n\} \right\} \chi_j(U_n) dU_n \quad (39)$$

and $\chi_j(U_n)$ is the characters of j irreducible representations. So using the character orthogonality

$$\begin{aligned} & \int \chi_j \left(U_n(\mathbf{x}, \tau) U_n(\mathbf{x}, \tau'')^\dagger \right) \chi_k \left(U_n(\mathbf{x}, \tau'') U_n(\mathbf{x}, \tau')^\dagger \right) dU_n(\mathbf{x}, \tau'') \\ &= \frac{\delta_{jk}}{d_j} \chi_j \left(U_n(\mathbf{x}, \tau) U_n(\mathbf{x}, \tau')^\dagger \right) \end{aligned} \quad (40)$$

and integrating over all $U_n(\mathbf{x}, \tau)$ variables except $U_n(\mathbf{x}, 0)$ and $U_n(\mathbf{x}, N_\tau - 1)$ one can obtain for $(N_\tau - 1)$ -section chains

$$\begin{aligned} & \int \exp \left\{ \frac{\beta}{N} \sum_{\tau=1}^{N_\tau-1} \text{ReSp} \left\{ U_n(\mathbf{x}, \tau) U_n^\dagger(\mathbf{x}, \tau-1) \right\} \right\} dU \\ &= \sum_j d_j \mathfrak{S}_j(\beta)^{N_\tau-1} \chi_j \left(U_n(\mathbf{x}, 0) U_n(\mathbf{x}, N_\tau - 1)^\dagger \right), \end{aligned} \quad (41)$$

which, as well as (16), differs from a one-section chain simply by $\mathfrak{S}_j(\beta) \rightarrow \mathfrak{S}_j(\beta)^{N_\tau-1}$.

Substituting (41) into (36) and taking into account that

$$\int \chi_j \left\{ \Omega(\mathbf{x}) U_n(\mathbf{x}, 0) \Omega^\dagger(\mathbf{x} + n) U_n^\dagger(\mathbf{x}, 0)^\dagger \right\} dU = \frac{\chi_j(\Omega(\mathbf{x})) \chi_j(\Omega(\mathbf{x})^\dagger)}{d_j} \quad (42)$$

we can integrate over $U_n(\mathbf{x}, N_\tau - 1)$ and finally get the familiar result [28, 29]

$$\exp \{ -\bar{S}_E \} = \sum_j \exp \left\{ -N_\tau \left(\lambda_G^{(j)}(\beta) + \ln \mathfrak{S}_0(\beta) \right) \right\} \chi_j(\Omega(\mathbf{x})) \chi_j(\Omega(\mathbf{x})^\dagger) \quad (43)$$

with⁷

$$\lambda_G^{(j)}(\beta) = \lambda_G^{(j)} \left(\frac{2N}{g^2} \right) \equiv -\ln \frac{\mathfrak{S}_j(\beta)}{\mathfrak{S}_0(\beta)}. \quad (44)$$

For the gauge groups $U(1)$ and $SU(2)$ we shall have $\mathfrak{S}_j(\beta) = I_j(\beta)$ and $\mathfrak{S}_j(\beta) = I_{2j+1}(\beta)$ correspondingly. Having made allowance for ([35]7.13.2(8))

⁷In $(1+1)$ -dimensional case we can integrate over $\Omega(\mathbf{x})$ and write for the partition function $Z = \sum_j \mathfrak{S}_j(\beta)^{N_\sigma N_\tau}$. This result is exact because in this case the action does not contain the magnetic part.

one can write

$$I_n(\beta) \approx \begin{cases} \frac{1}{\sqrt{2\pi\beta}} e^{\beta} \exp\{-\lambda n^2\}; & \beta \ll 1; \\ \frac{(\frac{\beta}{2})^n}{n!} \exp\left\{\left(\frac{1-\frac{1}{n}}{4n} - \frac{1}{18}\right) \beta^2\right\}; & \beta \gg 1. \end{cases} \quad (45)$$

Therefore in the *strong coupling region* ($\beta \gg 1$), we get the result which is very similar to that obtained for $Z(N)$

$$\lambda_G^{(j)}(\beta) = -\ln \frac{\mathfrak{S}_j(\beta)}{\mathfrak{S}_0(\beta)} \sim d_j \ln\left(\frac{2N}{\beta}\right). \quad (46)$$

Moreover, taking into account the results [37] obtained for $\int \chi_j^*(U) e^{\beta \chi} dU$, Eq. (46) can be easily generalized for $SU(N)$ case. Therefore, we may conclude that in the region $\beta \gg 1$ the effective coupling $\tilde{\beta}(N_\tau)$ depends on β and N_τ as

$$\tilde{\beta}(N_\tau) = F\left(\left(\frac{\beta}{const}\right)^{N_\tau}\right). \quad (47)$$

For the *weak coupling region* we can write

$$\lambda_G^{(j)}(\beta) \approx \lambda_N d_A \cdot \left[C_2(j) + O\left(\frac{j}{\beta}\right)^3 \right], \quad (48)$$

where $C_2(j)$ is the quadratic Casimir operator, $d_A = N^2 - 1$ for $SU(N)$ and $d_A = 1$ for $U(1)$. The function λ_G for $U(1)$ is given by

$$\lambda_1 = -\ln\left(\frac{I_1(\beta)}{I_0(\beta)}\right) \simeq \frac{1}{2\beta} + \left(\frac{1}{2\beta}\right)^2 \simeq \frac{g^2}{4} + \left(\frac{g^2}{4}\right)^2 \quad (49)$$

and that for $SU(2)$ by

$$\lambda_2 = \frac{1}{2\beta} + \left(\frac{1}{2\beta}\right)^2 + O(\beta^{-3}) = \frac{g^2}{8} + \left(\frac{g^2}{8}\right)^2 + O(g^6). \quad (50)$$

Similar, but less accurate result can be obtained for $SU(3)$ gauge group (with $j = \{l_1; l_2\}$). If we apply the asymptotic formula (45) to the well known expression [37]

$$\int \exp\left\{-\beta \frac{\chi}{3}\right\} \chi_{l_1 l_2} d\mu = \sum_n \det I_{l_j - j + i - n} \left(\frac{\beta}{3}\right) \quad (51)$$

this will give

$$\begin{aligned} & \int \exp \left\{ -\frac{\beta}{3} \text{ReSp}(U) \right\} \chi_{l_1 l_2} d\mu \\ & \approx \det I_{l_j - j + i} \left(\frac{\beta}{3} \right) e^{-\frac{8\lambda_3}{3}(l_1 + l_2)^2} \sum_n \exp \left\{ -8\lambda_3 \left(n + \frac{l_1 + l_2}{3} \right)^2 \right\}, \end{aligned} \quad (52)$$

with $l_2 > l_1 > 0$ and

$$\frac{8}{3}\lambda_3 = \frac{1}{\frac{2}{3}\beta} \left(1 + \frac{1}{\frac{2}{3}\beta} \right) + O(\beta^{-3}). \quad (53)$$

So, taking into account

$$\sum_n \exp \left\{ -8\lambda_3 \left(n - \frac{l_1 + l_2}{3} \right)^2 \right\} = \theta_3(8\lambda_3; 0) + O \left(\exp \left\{ -\frac{2\pi^2}{38\lambda_3} \right\} \right), \quad (54)$$

one can easily find

$$\begin{aligned} & \int \exp \left\{ -\frac{\beta}{3} \text{ReSp}(U) \right\} \chi_{l_1 l_2} d\mu \\ & \approx \sum_n \det I_{l_j - j + i - n} \left(\frac{\beta}{3} \right) \approx \exp \{ -8\lambda_3 \cdot C_2(l_1, l_2) \}, \end{aligned} \quad (55)$$

where $C_2(l_1; l_2) = \frac{1}{3}(l_1^2 + l_2^2 - l_1 l_2) - \frac{1}{3}$ is the quadratic Casimir operator for $SU(3)$ group and we immediately get an expression $\lambda_{SU(3)}^{(j)}(\beta)$ that formally coincides with (48). Therefore, the effective action is given by

$$\exp \{ -\bar{S}_E \} \approx [\mathfrak{I}_0(\beta)]^{N\tau} \sum_j e^{-8\lambda_3 N\tau \cdot C_2(j)} \chi_j(\Omega(\mathbf{x})) \chi_j(\Omega(\mathbf{x})^\dagger). \quad (56)$$

In the first order in $\frac{1}{\beta^2}$ the expression (56) coincides with that obtained in Hamiltonian approach developed in [38, 39]. It is worth to note that in [38, 39] the Hamiltonian gauge $U_0(\mathbf{x}, \tau) = 1$ can be imposed on all τ -links (in distinction from (36)) because there are no periodic border conditions. As a rule such condition is demanded within finite temperature theory [40], so at this point the approach of [38, 39] resembles zero temperature theory. However, as it is known from field theory [41], if we fix the gauge in a such way we would destroy the mechanism that guarantees the Gauss condition $Q^b = 0$ ($Q^b = (\delta^{bc} \partial_n - f^{abc} A_n^a) E_n^c$ for QFT and $Q^b = \sum_{n=-d}^d E_n^b$ for LGT) and a projection operator should be introduced to restore it.

In field theory, as well as in LGT [39] the integration variables $\phi_0^b(\mathbf{x})$ in the projection operator correspond to the parameters of static gauge transformations with the generators Q^b [41] and finally all looks as if the static gauge condition $U_0(\mathbf{x}, \tau) = U_0(\mathbf{x}) = \exp\{i\phi_0^b(\mathbf{x})T^b\}$ had been imposed from the beginning (as in (35)) instead of $U_0(\mathbf{x}, t) = 1$. Therefore, the auxiliary variables $\prod_{\tau=0}^{N_\tau} U_0(\mathbf{x}, \tau) = U_0(\mathbf{x})^{N_\tau} = \exp\{iN_\tau\phi_0^b(\mathbf{x})T^b\}$ in the projection operator formally correspond to Polyakov loops $\Omega(\mathbf{x})$ in (56) and the lattice length in the temporal direction $N_\tau a_\tau$ corresponds to the inverse temperature. So, at least in this particular case, the border conditions appear not to play a significant role.

Thus, we can use the results [36] obtained in Hamiltonian approach to sum over irreducible representation j in (43) which, e.g. for $SU(2)$ group, gives

$$\begin{aligned} & \sum_j e^{-j(j+1)\lambda} \cdot \frac{\sin\left[\frac{(2j+1)\varphi}{2}\right] \sin\left[\frac{(2j+1)\varphi'}{2}\right]}{\sin\frac{\varphi}{2} \sin\frac{\varphi'}{2}} \\ &= e^{-\frac{1}{4}N_\tau\lambda} \cdot \frac{\theta_3\left(\frac{\lambda}{2}; \frac{\varphi-\varphi'}{2}\right) - \theta_3\left(\frac{\lambda}{2}; \frac{\varphi+\varphi'}{2}\right)}{2 \sin\frac{\varphi}{2} \sin\frac{\varphi'}{2}}, \end{aligned} \quad (57)$$

and with good accuracy one can write

$$\begin{aligned} & \frac{\theta_3\left(\frac{\lambda}{2}; \frac{\varphi-\varphi'}{2}\right) - \theta_3\left(\frac{\lambda}{2}; \frac{\varphi+\varphi'}{2}\right)}{2 \sin\frac{\varphi}{2} \sin\frac{\varphi'}{2}} \\ & \approx \exp\left\{\tilde{\gamma} \cos\frac{\varphi}{2} \cos\frac{\varphi'}{2} + \delta \sin\frac{\varphi}{2} \sin\frac{\varphi'}{2} + G\right\}, \end{aligned} \quad (58)$$

where $\tilde{\gamma}\left(\frac{\lambda}{2}\right)$, $G\left(\frac{\lambda}{2}\right)$ and $\delta\left(\frac{\lambda}{2}\right)$ are smooth analytical functions of λ which are very similar to given in (31), and $\delta\left(\frac{\lambda}{2}\right)$, $G\left(\frac{\lambda}{2}\right) \ll \tilde{\gamma}\left(\frac{\lambda}{2}\right)$. Then we can finally write

$$\exp\{-\bar{S}_E\} \approx \exp\left\{\tilde{\beta}(N_\tau) \chi_j(\Omega(\mathbf{x})) \chi_j(\Omega(\mathbf{x})^\dagger) + const\right\}, \quad (59)$$

with

$$\tilde{\beta}(N_\tau) \equiv \tilde{\gamma}\left(\frac{2N_\tau}{\beta}\right) \approx \begin{cases} 2 \exp\left\{-\frac{2N_\tau}{\beta}\right\}; & N_\tau \gg \beta \gg 1; \\ \frac{\beta}{N_\tau}; & \beta \gg N_\tau. \end{cases} \quad (60)$$

In similar manner one may show that in the case of $SU(3)$ gauge group effective coupling dependence on β and N_τ resembles (60) and can be written

as

$$\tilde{\beta}(N_\tau) \approx f(\lambda_G N_\tau) \quad (61)$$

with λ_G given by (49), (50) and (53).

We may conclude that in given approximation for continuous abelian $U(1)$ and nonabelian groups $SU(2)$ and $SU(3)$ the effective couplings $\tilde{\beta}(N_\tau)$ are also totally determined by one-dimensional chains, however, contrary to $Z(N)$ case (33) $\tilde{\beta}(N_\tau)$ is changing essentially slower with the bare coupling β .

In Appendices A and B we made an attempt to perturbatively estimate the contribution of the magnetic part of the action. First order correction in magnetic coupling β_σ turns into zero for symmetry reasons. The most important (from our viewpoint) part of the second order correction $\beta_\sigma^2 \Xi_2$ leads to simple renormalisation of the effective coupling and at least in a limit $N_\tau \rightarrow \infty$ does not change the results drastically.

More complicated (and possibly more accurate) expression for $\beta_\sigma^2 \Xi_2$ was suggested in [30]. As it was established in this paper the additional term $\beta_\sigma^2 \Xi_2$ makes the movement of T_c with increasing N_τ slower [30]. Unfortunately, it is not so easy to anticipate whether it can totally terminate the dependence of T_c on N_τ , especially on extremely large lattices, because the perturbative expansion in β_σ is hardly applicable outside of the narrow area $\beta_\sigma^2 N_\tau \sim \frac{1}{T} \frac{a_\tau}{g_\sigma^4 a_\sigma^2} \ll 1$.

Moreover, if following [30] we assume that second order corrections in β_σ radically change the predictions obtained from zero order, the convergence of series in β_σ should be reconsidered and one must prove that the corrections of orders higher than the second one are insignificant. Anyway perturbative series can hardly be regarded as a reliable instrument for the computation of effective action in the area $\beta_\sigma \ll 1$, $N_\tau \rightarrow \infty$ (see also footnote 4)

As it has been already pointed out, $\beta_\sigma = \frac{2N}{g^2 \xi}$ can be made negligibly small in comparison with $\beta_\tau = \frac{2N}{g^2 \xi}$ by changing the anisotropy parameter ξ . Precise analysis of modifications of $\beta_{\tau,\sigma}$ with changing ξ has been made by F. Karsch in [44]:

$$\beta_\tau = \xi(\bar{\xi} - 0.27192) + \frac{1}{2}; \quad \beta_\sigma = \frac{\bar{\xi} + 0.39832}{\xi}; \quad \bar{\xi} = \sqrt{\beta_\tau \beta_\sigma}. \quad (62)$$

However, this going into details cannot change the situation drastically for large ξ (Hamiltonian limit) and large β (continuum limit). In this particular area one can hardly hope that the second order corrections in β_σ will play such a significant role.

4 'Electric toy' model

Since the pioneering work of Svetitsky and Yaffe [42], it is understood that all the relevant properties of the phase structure can be encoded in a suitable effective action for the order parameter, e.g. the Polyakov loop. The simplest and most popular approach to such an action was suggested in [28, 29]. We use the anisotropic lattice ($a_\tau \ll a_\sigma$), that helps to avoid the strong coupling approximation (one of the basic assumption in [28, 29]). Then we try to trace a_τ -dependance of the effective coupling and compute Callan-Symanzik beta function for $N_\tau \rightarrow \infty$.

As it is pointed out in [42] at high temperatures $T_{SY} \propto \beta_\sigma \propto 1/T_{SY}$ and $\beta_\tau \propto T_{SY}$ (in our denotation $\beta_\sigma \propto 1/\xi$ and $\beta_\tau \propto \xi$). Consequently, the functional integral is highly peaked at the configurations in which the spatial link variables are static up to gauge transformations and the system is described by zero temperature d -dimensional gauge theory with coupling $g^2 T_{SY}$. The dynamics of the spatial gauge fields is regarded [42] as being qualitatively the same at high and low temperatures.

The naive interpretation of this suggestion might be as follows: the magnetic part of the action S_M , proportional to $\beta_\sigma \propto 1/\xi$ do not play an essential role at high temperatures and the effective action, obtained by integration over spatial degrees of freedom should not be influenced by the magnetic part in the vicinity of the Hamiltonian limit ($\xi \rightarrow \infty$). However, there are serious reasons to believe that the magnetic part of the action may be of crucial importance for creating the confining forces [45, 46, 47], even at high temperatures [48].

Our computation reveals no drastic changes resulting from lower corrections in β_σ (see Appendices A and B), however, one would hardly suggest with certainty that the picture will be the same for the whole series in β_σ . A simple example would prove that such suggestion may be erroneous. Indeed, let us consider $d=2$ - Ising model on the anisotropic lattice. Free energy density may be written as

$$F(\gamma_\sigma; \gamma_\tau) = -\frac{\ln Z}{N_\sigma N_\tau} = \ln \frac{(1 - \gamma_\sigma)(1 - \gamma_\tau)}{2} + \int_0^{2\pi} \frac{d\varphi_\tau d\varphi_\sigma}{(2\pi)^2} \ln f \quad (63)$$

where

$$f = \left(1 + \gamma_\sigma^2\right) \left(1 + \gamma_\tau^2\right) - 2\gamma_\tau \left(1 - \gamma_\sigma^2\right) \cos \varphi_\tau - 2\gamma_\sigma \left(1 - \gamma_\tau^2\right) \cos \varphi_\sigma. \quad (64)$$

and

$$\gamma_{\sigma,\tau} = \tanh \beta_{\sigma,\tau}. \quad (65)$$

⁸In [42] temperature T_{SY} is defined as $T_{SY} \equiv \sqrt{\xi_\tau/\xi_\sigma}/N_\tau a_\tau$

It is evident that for any arbitrary small β_σ there is a critical value of β_τ

$$\beta_\tau^c = -\frac{1}{2} \ln \tanh \beta_\sigma \quad (66)$$

at which the system undergoes a phase transition. Although perturbative expansion in β_σ gives a reasonable value of F , the finite series in β_σ do not reproduce the singularity of F at $\beta_\tau = \beta_\tau^c$. In such simple case, the reason of the above can be easily traced. The main contribution into integrand at (63) gives the area of $\varphi_{\sigma,\tau} \rightarrow 0$, so

$$\ln f = \ln(1 + \gamma_\tau - 2\gamma_\tau \cos \varphi_\tau) + v\gamma_\sigma - \frac{1}{2}\gamma_\sigma^2 v^2 + O(\gamma_\sigma^3) \quad (67)$$

with

$$v = \frac{1 + \gamma_\tau + 2\gamma_\tau \cos \varphi_\tau - 2(1 - \gamma_\tau) \cos \varphi_\sigma}{1 + \gamma_\tau - 2\gamma_\tau \cos \varphi_\tau} \mathbf{v} \frac{1 + 5\gamma_\tau}{1 - \gamma_\tau} \quad (68)$$

hence, in the area of $\gamma_\tau \sim 1$ the expansion parameter of free energy F is, indeed, $\frac{6}{1-\gamma_\tau}\gamma_\sigma$ rather than γ_σ .

In the considered example, the restoration of real physical picture with extending perturbative series in β_σ is very slow. For this reason, one should be careful when cutting the series in β_σ in gluodynamics; at some stage the magnetic part contribution may start dominating and might completely change the final result. Therefore, *QCD* without the magnetic part should be considered as a specific model.

An effective action (43) of such *electric toy model* differs from that obtained in [28, 29] only by initial assumption: we do not use strong coupling approximation and work on the anisotropic lattice. An additional assumption that the magnetic part may be discarded in the limit $\xi \rightarrow \infty$, give us the right to study weak coupling region and consider the limit $a_\tau \rightarrow 0$ in such model. It is evident, that at the same time we can demand $a_\sigma \rightarrow 0$ and $a_\tau N_\tau = T^{-1} = \text{const} < \infty$.

Although (43) was computed exactly along the line of Svetitsky-Yaffe program, discarded magnetic part of the action may lead to serious distortions. Therefore, it is interesting to compare (43) with the analytical results obtained without such approximation. In [42] the analytical solution was found for $(2+1)$ -dimensional $U(1)$ -gluodynamics and a relation was established between $(2+1)$ -dimensional $U(1)$ -gluodynamics and $2-d$ Coulomb Gas model (CGM), which undergoes a Berezinskii-Kosterlitz-Thouless phase transition at the point of $2\pi^2 \frac{\beta}{N_\tau} = \beta_c^{CGM} \approx \frac{2}{\pi}$.

The model action (43) connects $(2+1)$ -dimensional $U(1)$ -gluodynamics with $2d$ -XY model. As it is known, the 2D X -Y model can be mapped to

the 2D Coulomb gas (for recent review see [49]). Therefore, in exact [42] and approximate approach one comes to similar description of the phase structure (2 + 1)- dimensional $U(1)$ -gluodynamics. The only thing which remains is to compare the prediction of [42] for the critical coupling $\beta_c \approx \frac{4}{\pi T} \approx \frac{1.2732}{T}$ (in $a_\tau = 1$ units) with that one which follows for $N_\tau \gg \beta$ from (43), that is with $\beta_c \approx \frac{\beta_c^{BKT}}{T}$ (in the same units), where β_c^{BKT} is the critical coupling of Berezinskii-Kosterlitz-Thouless phase transition. Mote Carlo experiment [50] gives $\beta_c^{BKT} \approx 1.11$. For the *toy* model, we cannot expect more.

5 Effective coupling and critical temperature

The effective action (18) for $Z(N)$ - gluodynamics has the critical point at $\tilde{\beta}(N_\tau) = \tilde{\beta}_c(N_\tau)$ where $\tilde{\beta}_c(N_\tau)$ are numeric constants and the critical point will depend only on N and the space dimension. If we assume that this critical point does correspond to the temperature phase transition, then as it follows from (21) this will happen at the fixed temperature T_c (in the strong coupling regime) when

$$\left(\frac{\beta}{const}\right)^{N_\tau} = const, \quad (69)$$

then by the appropriate choice of constants one can obtain scaling having imposed the condition (5), but as for the string tension the scaling is hardly possible under (4) condition.

In the *weak coupling* area, in accordance with (33), and (24) we may put

$$\tilde{\beta}(N_\tau) \approx \begin{cases} \exp\{-4N_\tau e^{-2\beta}\}; & N = 2; \\ \exp\{-3N_\tau e^{-\beta\frac{3}{2}}\}; & N = 3; \\ \exp\left\{-N_\tau \left(\frac{2\pi}{N}\right)^2 \exp\left\{-\frac{\beta}{2} \left(\frac{2\pi}{N}\right)^2\right\}\right\}; & N \gg 1. \end{cases} \quad (70)$$

so as it follows from

$$\begin{aligned} -\ln \tilde{\beta}(N_\tau) &\approx N_\tau \left(\frac{2\pi}{N}\right)^2 e^{-\frac{\beta}{2} \left(\frac{2\pi}{N}\right)^2} \\ &= N_\tau \left(\frac{2\pi}{N}\right)^2 (\Lambda a_\tau)^{\xi \frac{b_0(4\pi^2)}{N}} \approx -\ln \tilde{\beta}_c \approx const, \end{aligned} \quad (71)$$

the critical point (e.g. for $N \gg 1$) will escape undesirable dependance on N_τ if we demand

$$\frac{\beta}{2N} \equiv g_\tau^{-2} \approx b_0 \ln \frac{1}{\Lambda a_\tau}; \quad \Lambda = \text{const} \quad (72)$$

where

$$b_0 \approx \frac{N}{4\pi^2} \approx .025N, \quad (73)$$

which for $\xi = 1$ and N gives good agreement with $b_0 = \frac{11}{12} \frac{N}{4\pi^2} \approx 0.023N$. Therefore, we may conclude that

$$T_c \simeq \text{const} \times a_\tau^{-1+\xi} \frac{b_0(4\pi^2)}{N} \simeq \text{const} \times a_\tau^{-1+\xi}. \quad (74)$$

As it has been shown [27]

$$\sqrt{\sigma} \approx \frac{a^{-1}}{\sqrt{2}} \exp \left\{ -\beta \left(1 - \cos \frac{2\pi}{N} \right) \right\} \approx \frac{a^{-1}}{\sqrt{2}} \exp \left\{ -\frac{4\pi^2}{N} b_0 \ln \frac{1}{\Lambda a_\tau} \right\} \quad (75)$$

and (74) at given approximation the scaling will take place or, in other words, we shall have

$$\frac{\sqrt{\sigma}}{T_c} = (a_\tau)^{\frac{b_0(4\pi^2)}{N}(\xi - \frac{1}{\xi})} \cdot \text{const}. \quad (76)$$

In principle, by changing ξ one can tune the parameter b_0 to make T_c in (74) or $\sqrt{\sigma}$ in (75) tend to the finite value in the continuum limit, however, at least, under given approximation it cannot be done simultaneously and the ratio $\frac{\sqrt{\sigma}}{T_c}$ will be independent from lattice spacing only for $\xi = 1$.

6 Continuous groups

In this case the requirement that the position of critical point $\tilde{\beta}(N_\tau) = \tilde{\beta}_c$ corresponding to the temperature $T = T_c$ of phase transition should be independent of lattice sizes, (at least for $N_\tau \rightarrow \infty$) leads to the condition

$$N_\tau \lambda_G \rightarrow \text{const}. \quad (77)$$

For large enough β (small a_τ) $\lambda_{zG} \sim \frac{1}{2\beta}$, so (77) means trivial asymptotic freedom

$$\frac{1}{2N} \lim_{a_\tau \rightarrow 0} \frac{g^2}{a_\tau} \equiv T_N = \text{const}. \quad (78)$$

For example, in the case of $SU(2)$ gauge group from (77) and (60) we obtain the condition

$$\tilde{\beta}(N_\tau) \approx \tilde{\gamma} \left(\frac{N_\tau}{N} g^2 \right) \approx \tilde{\gamma} \left(\frac{1}{N} \frac{T_N}{T_c} \right) \approx \tilde{\beta}_c \quad (79)$$

and consequently

$$T_N = T_c N \tilde{\gamma}^{-1} \left(\tilde{\beta}_c \right), \quad (80)$$

where $\tilde{\gamma}^{-1}(x) = y$ is the function inverse to $\tilde{\gamma}(y) = x$, so $\tilde{\gamma}^{-1}(\tilde{\beta}_c)$ is simply a numeric constant.

Now we may write for the effective coupling

$$\tilde{\beta}(N_\tau) \approx \tilde{\gamma} \left(\tilde{\gamma}^{-1} \left(\tilde{\beta}_c \right) \frac{T_c}{T} \right) \quad (81)$$

with

$$\tilde{\gamma} \left(\tilde{\gamma}^{-1} \left(\tilde{\beta}_c \right) t \right) \approx \begin{cases} 2 \exp \left\{ -t \tilde{\gamma}^{-1} \left(\tilde{\beta}_c \right) \right\}; & t \gg 1; \\ \frac{t}{2 \tilde{\gamma}^{-1} \left(\tilde{\beta}_c \right)}; & t \ll 1. \end{cases} \quad (82)$$

For the calculation of temporal string tension α , higher orders in a_τ are needed in the expansion of g^2 . Indeed, to find α we may compute the correlation function $\langle \chi_0 \chi_R \rangle$ between two probes⁹

$$\langle \chi_0 \chi_R \rangle = \int \frac{\exp \{ i q R \}}{s_0 - \gamma \sum_n \cos q_n} \left(\frac{dq}{2\pi} \right)^3; \quad \gamma = \frac{g^2}{2aT}. \quad (83)$$

That can be done in spherical model approximation in the same way as in [36]. The saddle point s_0 is defined from the condition $\langle \chi_0 \chi_R \rangle = 1$ and is equal to

$$s_0 \approx \left\{ \begin{array}{l} 3\gamma_c + 8\pi^2 \gamma_c (\gamma_c - \gamma)^2; \\ \gamma \cdot \gamma_c. \end{array} \right. \quad (84)$$

For $R \gg 1$ one may write

$$\langle \chi_0 \chi_R \rangle \approx (2\pi R)^{-\frac{3}{2}} \exp \{ -\alpha R \} \quad (85)$$

and for string tension α we get

$$\alpha \sim (\gamma_c - \gamma) \cdot \theta(\gamma_c - \gamma) \quad (86)$$

or

$$\alpha \approx \frac{g^2(T) - g^2(T_c)}{2a^2T} \theta(T_c - T). \quad (87)$$

⁹Distance R is measured in lattice units.

If we claim the independence of α from lattice sizes then we should demand:

$$g^2(T) \approx \left\{ -4Ta \ln \left(\exp \left\{ -\frac{g^2(T_c)}{4Ta} \right\} - \frac{\alpha}{2}a \right) \right\}; \quad T < T_c. \quad (88)$$

This expression guarantees scaling for α in given approximation. However, if in continuum limit $\alpha \rightarrow \text{const}$ then, at least, the second order expansion g^2T in a is needed and, consequently, in this case we must compute $\beta_{CS}(g^2)$ in the higher orders in g . An example of a such computation is given in the Appendix C.

As it is seen from Eq. (117), $\beta_{CS}(g^2) \sim -\frac{g}{2}$ when $g \rightarrow 0$, which strongly contradict to standard expression (1), where $\beta_{CS}(g^2) \sim -b_0 g^3$ in corresponding area. It looks so as if for extremely small a (and consequently extremely small $g^2(a)$) the 'recursive' coupling $g^2(\lambda a) \equiv F(g^2(a); \lambda)$ might be presented as

$$g^2(\lambda a) \simeq c_0(\lambda) + c(\lambda) \cdot (g^2(a))^\alpha + \dots \quad (89)$$

where α is not obligatory an integer, but is *independent* of λ . Asymptotic freedom leads to $c_0(\lambda) = 0$, so we immediately get

$$\beta_{CS}(g) = a \frac{\partial g}{\partial a} \simeq \frac{c(1)}{c'(1)} g + O(g^3). \quad (90)$$

Although the phase structure of the system defined by the action (35) and (43) essentially depends upon space dimension d , the effective coupling is not sensitive to it and is defined by one-dimensional chain. Therefore, it is not too surprising that the toy model shows *trivial* asymptotic freedom inherent to Schwinger model¹⁰.

7 Conclusions

Although in a strong coupling region non-universal behavior of $\beta_{CS}(g)$ is observed, in a weak coupling limit we find the dependence of g^2 on lattice spacing for $Z(2)$ and $Z(3)$ gauge groups. This dependence is very close to that obtained in standard renormalisation theory. However, the dependence on lattice anisotropy parameter ξ is too strong compared with [44] and differs for spatial string tension and critical temperature. Therefore, either an additional procedure must be added to remove the remaining ξ -dependence in physical values or one must work only at $\xi = 1$. In the last case our results should be reconsidered, because we worked far from the area of $\xi \simeq 1$.

¹⁰ Another cause of similarity is the absence of magnetic field in Schwinger model.

Main approximation, used to obtain an effective action for continuous groups is that in a Hamiltonian limit ($\xi \gg 1$) the magnetic part of the action, (proportional to $1/\xi$) can be neglected compared to the electric one (proportional to ξ). Perturbative estimation of the magnetic part does not change the results considerably. Therefore, in the limit of $N_\tau \rightarrow \infty$ and $a_\tau \rightarrow 0$ the toy model shows the trivial asymptotic freedom ($g^2 \sim a_\tau$). This strongly contradicts the results obtained in the perturbation theory.

Another noteworthy point is to be underscored: a considerable difference between $Z(N)$ and continuous groups claimed by equations (72) and (117). It looks somewhat ridiculous so we would like to discuss it in more detail. We argue that such difference between $Z(N)$ and $U(1)$ will not disappear for any large (but finite) N and will be washed out only at $N \rightarrow \infty$, because on their way to continuum limit ($\beta \rightarrow \infty$) $Z(N)$ groups inevitably pass the point of $\beta \mathbf{v} \frac{N^2}{(2\pi)^2} = \text{const}$. With further decreasing a the bare coupling $\beta \gg \frac{N^2}{(2\pi)^2}$ and, consequently, effective coupling $\tilde{\beta}(N_\tau)$ exponentially decrease with β , as it can be seen from (24) which finally leads to (117), so the dependence of g^2 on a is similar to standard and leads to nontrivial asymptotic freedom. In the case of $U(1)$ gauge group (which corresponds to $N = \infty$) the point of $\beta \mathbf{v} \frac{N^2}{(2\pi)^2}$ is evidently unreachable. Therefore, we get different results if the limit $N \rightarrow \infty$ is taken before the computation of partition function, and after that, in other words, the limits $N \rightarrow \infty$ and $N_\tau \rightarrow \infty$ do not 'commutate'.

Finally we want to stress again that despite the gluodynamics without a magnetic part is very far from reality, it may be interesting not only for simplicity reasons, but also as an example of a model where the renormalisation procedure may be fulfilled analytically and exclusively within the lattice gauge theory. At the same time, it is a good laboratory to study the nature of some non-perturbative phenomena in LGT revealing the difference between the models with discrete and continuous gauge groups and it is not unlikely that this difference will be preserved in more realistic models.

8 Appendix A. Perturbation series in $\beta_\sigma \equiv \frac{2N}{g^2\xi}$ for $Z(2)$ gauge group.

Now, for simplicity, we shall confine ourself to the case of $Z(2)$ gauge group and, in fact, use the series in $\gamma_\sigma \equiv \tanh \beta_\sigma$:

$$Z(\Omega) = Z_0(\Omega) \cdot \sum_{n=0}^{\infty} \gamma_\sigma^n \Xi_n(\Omega) \quad (91)$$

The zero order term $Z_0(\Omega)$ has been already considered. It is easy to see that the first order term Ξ_1 is equal to zero. Only the pairs of plaquettes having equal spatial coordinates and orientations, but positioned at different temporal points τ and $\tau + \Delta$ along the temporal axis contribute into the second order Ξ_2 . Each of the four chains $1 + \Omega(\mathbf{x}) \Omega(\mathbf{x} + \mathbf{n}) \gamma^{N_\tau}$ which cross pair links $z_n(\tau, \mathbf{x})$ and $z_n(\tau + \Delta, \mathbf{x})$ of these plaquettes is converted into $\gamma^\Delta + \Omega(\mathbf{x}) \Omega(\mathbf{x} + \mathbf{n}) \gamma^{N_\tau - \Delta}$.

If we denote

$$\begin{aligned} \Omega(\mathbf{x}) &= \Omega_1; & \Omega(\mathbf{x} + \mathbf{n}) &= \Omega_2; \\ \Omega(\mathbf{x} + \mathbf{n} + \mathbf{m}) &= \Omega_3; & \Omega(\mathbf{x} + \mathbf{m}) &= \Omega_4 \end{aligned} \quad (92)$$

and

$$\begin{aligned} \hat{\mathbf{I}} &\equiv \frac{\Omega_1\Omega_2 + \Omega_2\Omega_3 + \Omega_3\Omega_4 + \Omega_4\Omega_1}{4}; \\ \hat{\mathbf{I}}^2 &= \frac{1 + \Omega_1\Omega_3 + \Omega_2\Omega_4 + \Omega_1\Omega_2\Omega_3\Omega_4}{4} \end{aligned} \quad (93)$$

and take into account that $\hat{\mathbf{I}} = \hat{\mathbf{I}}^3$ and

$$\prod_{n=1}^4 (p + q\Omega_n\Omega_{n+1}) = (p^2 - q^2)^2 + 4pq(p^2 + q^2)\hat{\mathbf{I}} + 8p^2q^2\hat{\mathbf{I}}^2 \quad (94)$$

after bulky computations we get [31]:

$$\Xi_2(\Omega) = N_\tau \sum_{\Delta=1}^{N_\tau-1} \prod_{n=1}^4 \frac{\gamma^\Delta + \Omega_n\Omega_{n+1}\gamma^{N_\tau-\Delta}}{1 + \Omega_n\Omega_{n+1}\gamma^{N_\tau}} \equiv N_\tau^2 (Q_0 + 2Q_1\hat{\mathbf{I}} + Q_2\hat{\mathbf{I}}^2) \quad (95)$$

where Q_j are the functions of γ and N_τ . If e.g. for $Z(2)$ we denote

$$\gamma = \exp\{-2e^{-2\beta}\} = \exp\{-2(a_\tau\Lambda)^{8\tilde{b}_0}\} \quad (96)$$

then for $8\tilde{b}_0 \approx 1$ it gives $\gamma^{N_\tau} \approx e^{-\varepsilon}$ with $\varepsilon = \frac{2\Lambda}{T}$ and for $N_\tau \gg 1$ we obtain

$$Q_0 \approx \frac{\sinh 2\varepsilon - 2\varepsilon}{4\varepsilon \sinh^2 \varepsilon} \simeq \begin{cases} \frac{1}{3} - \frac{2}{45}\varepsilon^2; & \varepsilon \ll 1; \\ \frac{1}{2\varepsilon}; & \varepsilon \gg 1; \end{cases} \quad (97)$$

$$Q_1 \approx \frac{(\cosh 2\varepsilon + 5) \sinh \varepsilon - 6\varepsilon \cosh \varepsilon}{4\varepsilon \sinh^4 \varepsilon} \simeq \begin{cases} \frac{1}{5} - \frac{17\varepsilon^2}{210}; & \varepsilon \ll 1; \\ \frac{1}{\varepsilon} e^{-\varepsilon}; & \varepsilon \gg 1; \end{cases} \quad (98)$$

$$Q_2 \approx \frac{(2\varepsilon \cosh \varepsilon - 3 \sinh \varepsilon) \cosh \varepsilon + \varepsilon}{8\varepsilon (\sinh \varepsilon)^4} \simeq \begin{cases} \frac{1}{30} - \frac{\varepsilon^2}{63}; & \varepsilon \ll 1; \\ \frac{1}{4} e^{-2\varepsilon}; & \varepsilon \gg 1. \end{cases} \quad (99)$$

The term $\hat{\mathbf{I}}$ contains only the nearest neighbor interactions of Ω_n and leads to a simple shift of effective coupling. Although the term $\hat{\mathbf{I}}^2$ also includes interactions, that are absent in the standard Ising model, such modifications has been extensively studied and it was shown (see e.g.[51]) that they might change the phase structure so drastically that this theory would not belong any more to the same universality class as the ordinary one. However, if the corresponding coupling $\tilde{\beta}(N_\tau) - \beta(N_\tau)$ is less than $\tilde{\beta}(N_\tau)/6$ (as it certainly is in our case) such interactions does not bring sufficient changes into the critical behavior of partition function [51]. Taking into account that only in the 1/8 part of the configurations $(\Omega_n = -\Omega_{n+1})$, $\hat{\mathbf{I}}$ differs from $\hat{\mathbf{I}}^2$ we put $Q_0 + 2Q_1\hat{\mathbf{I}} + Q_2\hat{\mathbf{I}}^2 \approx Q_0 + (2Q_1 + Q_2)\hat{\mathbf{I}}$. It is easy to check that in such approximation $K + \varepsilon\hat{\mathbf{I}} \approx K \exp\left(\frac{\varepsilon}{K} \cdot \hat{\mathbf{I}}\right)$ so after the inclusion of the second order correction we finally get

$$\begin{aligned} \frac{Z(\Omega)}{Z_0(\Omega)} &\approx \left(1 + \gamma_\sigma^2 N_\tau^2 Q_0\right) \left(1 + \frac{\gamma_\sigma^2 N_\tau^2 (2Q_1 + Q_2)}{1 + \gamma_\sigma^2 N_\tau^2 Q_0} \hat{\mathbf{I}} + \dots\right) \\ &\approx \exp\left(\tilde{\beta}_{(2)}(N_\tau) \hat{\mathbf{I}} + \text{const}\right) \end{aligned} \quad (100)$$

with

$$\tilde{\beta}_{(2)}(N_\tau) = \frac{2Q_1 + Q_2}{(\gamma_\sigma N_\tau)^{-2} + Q_0} \quad (101)$$

which evidently leads to a simple shift in effective coupling.

9 Appendix B. Perturbation series in β_σ for $U(1)$ - gauge group

The zero order term $Z_0(\Omega)$ of the expansion

$$Z(\Omega) = Z_0(\Omega) \cdot \left(1 + \beta_\sigma \Xi_1(\Omega) + \frac{\beta_\sigma^2}{2} \Xi_2(\Omega) + \dots\right) \quad (102)$$

corresponds to the case $\beta_\sigma = 0$ already considered. It is easy to see that the first order term Ξ_1 is equal to zero. Computation of the second order term: Ξ_2 is very similar to that for discrete groups [31], but technical difficulties increase enormously. The procedure suggested here differs from that elaborated in [30] chiefly in technicalities. We are forced to partly sacrifice accuracy to obtain a result which appears to us rather simple and transparent.

Only the pairs of plaquettes having equal spatial coordinates and orientations contribute into the second order term: Ξ_2 but they are positioned at different points τ and $\tau + \Delta$ along the temporal axis.

Such additional plaquettes convert one-dimensional chains

$$\sum_j [I_j(\beta)]^{N_\tau} e^{ij(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\mathbf{n}})}, \quad (103)$$

which cross the pair links $U_n(\tau, \mathbf{x})$ and $U_n(\tau + \Delta, \mathbf{x})$ of these plaquettes into

$$\sum_j [I_{j+1}(\beta)]^\Delta [I_j(\beta)]^{N_\tau - \Delta} e^{ij(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\mathbf{n}})}, \quad (104)$$

so we get

$$\Xi_2 = \frac{N_\tau \sum_{jlr s} \Theta_{jrls}}{\sum_{jlr s} \left(\frac{I_j(\beta) I_r(\beta) I_l(\beta) I_s(\beta)}{I_0(\beta)^4} \right)^{N_\tau} \cos(\phi_{j,r,l,s})} \quad (105)$$

where

$$\Theta_{jrls} = 2 \cos(\phi_{j,r,l,s}) \sum_{\Delta=0}^{\frac{N_\tau}{2}-1} \left(\frac{I_{j+1} I_{r+1} I_{l-1} I_{s-1}}{I_0^4} \right)^\Delta \left(\frac{I_j I_r I_l I_s}{I_0^4} \right)^{N_\tau - \Delta} \quad (106)$$

and

$$\begin{aligned} \phi_{jrls} = & j(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\mathbf{n}}) + r(\varphi_{\mathbf{x}+\mathbf{n}} - \varphi_{\mathbf{x}+\mathbf{n}+\mathbf{m}}) + \\ & l(\varphi_{\mathbf{x}+\mathbf{n}+\mathbf{m}} - \varphi_{\mathbf{x}+\mathbf{m}}) + s(\varphi_{\mathbf{x}+\mathbf{m}} - \varphi_{\mathbf{x}}). \end{aligned} \quad (107)$$

The term Ξ_2 may be easily computed for $\frac{N_\tau}{2\beta} \gg 1$ (e.g., $N_\tau \sim \frac{1}{a_\tau}$; $\beta \sim \ln \frac{1}{a_\tau}$; $a_\tau \rightarrow 0$). In such area, taking into account (45), leading terms may be written as

$$\Theta_{0000} = \Theta_{-1-111} = 2 \frac{1 - e^{-\frac{2N_\tau}{\beta}}}{1 - e^{-\frac{4}{\beta}}} \quad (108)$$

The next terms

$$\begin{aligned}\Theta_{(1)} &= 2(\Theta_{-1000} + \Theta_{0-100} + \Theta_{0010} + \Theta_{0001}) \\ &\simeq 2 \frac{e^{-\frac{N_\tau}{\beta}}}{1 - e^{-\frac{2}{\beta}}} \sum_k \cos(\varphi_k - \varphi_{k+1}) + O\left(e^{-2\frac{N_\tau}{\beta}}\right)\end{aligned}\quad (109)$$

are of order $e^{-\frac{N_\tau}{\beta}}$. Finalizing we may write

$$\Theta_{0000} + \Theta_{(1)} \simeq \frac{2}{1 - e^{-\frac{4}{\beta}}} \left(1 - e^{-\frac{N_\tau}{\beta}} \left(1 - e^{-\frac{2}{\beta}}\right) \sum_k \cos(\varphi_k - \varphi_{k+1})\right) \quad (110)$$

and

$$1 + \frac{\beta_\sigma^2}{2} \Xi_2 \simeq \left(1 + \frac{\beta_\sigma^2 N_\tau}{1 - e^{-\frac{4}{\beta}}}\right) \exp\left(-\tilde{\beta}_{(2)} \sum_k \cos(\varphi_k - \varphi_{k+1})\right) + O\left(e^{-2\frac{N_\tau}{\beta}}\right) \quad (111)$$

Therefore, for the effective action we get

$$-S_{eff}(\beta; \beta_\sigma) \approx \left(\tilde{\beta} - \tilde{\beta}_{(2)}\right) \sum_{\mathbf{x}, \mathbf{n}} \cos(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\mathbf{n}}) + O\left(\beta_\sigma^4\right), \quad (112)$$

where

$$\tilde{\beta}_{(2)} = e^{-\frac{N_\tau}{\beta}} \left(1 - e^{-\frac{2}{\beta}}\right) \left(1 + \frac{1 - e^{-\frac{4}{\beta}}}{\beta_\sigma^2 N_\tau}\right)^{-1}. \quad (113)$$

Corrections of higher order in $e^{-\frac{N_\tau}{\beta}}$ include 'abnormal' terms like $\cos(\varphi_{\mathbf{x}} - \varphi_{\mathbf{x}+\mathbf{n}+\mathbf{m}})$ similar to $\hat{\mathbf{I}}^2$ in (95). If series in β_σ converge, corrections in β_σ are of little importance in the area of $N_\tau \gg \beta$.

10 Appendix C

We are interested in the limit $N_\tau \rightarrow \infty$, therefore, if there exists such β_{asympt} that for all $\beta > \beta_{asympt}$ then we shall have $\lambda_G^{(j)} > \lambda_G^{(j_0)} = \lambda_{\min}$. Therefore, for $T < T_c$ ($N_\tau \gg \lambda_{\min}$) we may discard all terms in (43) except those corresponding to $j = 0$ and j_0 . As it can be shown $\beta_{asympt} \simeq 5/3$ ($g_{asympt} \simeq 1$) for $U(1)$ and $\beta_{asympt} \simeq 7/4$ ($g_{asympt} \simeq 3/2$) for $SU(2)$ gauge group. In all cases, which we consider, j_0 corresponds to fundamental representation, so we can preserve only two first terms in (43) and may write

$$-\bar{S}_E \approx N_\tau \ln \mathfrak{S}_0(\beta) + \exp\{-N_\tau \lambda_G(\beta)\} \chi(\Omega(\mathbf{x})) \chi(\Omega(\mathbf{x})^\dagger). \quad (114)$$

In such a case instead of (45) one may use ([35] 7.13.2(5))

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \sum_{m=0}^{M-1} (-2x)^{-m} \frac{\Gamma\left(\frac{1}{2} + \nu + m\right)}{\Gamma\left(\frac{1}{2} + \nu - m\right) \Gamma(1 + m)} + O(x)^{-M} \quad (115)$$

for more accurate ¹¹ computation of beta function, which, taking into account (77), may be written as

$$\beta_{CS}(g^2) \equiv -a_\tau \frac{\partial g}{\partial a_\tau} = -a_\tau \left(\frac{\partial a_\tau}{\partial g} \right)^{-1} = - \left(\frac{\partial}{\partial g} \ln \lambda_G \right)^{-1}. \quad (116)$$

In particular for $G = SU(2)$

$$\beta_{CS}(g^2) \approx -\frac{g}{2} \left(1 - \frac{g^2}{8} + \left(\frac{g^2}{8} \right)^2 + \frac{17}{2} \left(\frac{g^2}{8} \right)^3 + 47 \left(\frac{g^2}{8} \right)^4 \right) + 273 \left(\frac{g^2}{8} \right)^5. \quad (117)$$

It is easy to check that if we preserve only two first terms in (117) the beta function shows dip at $g_{dip} = \sqrt{8}$ (non-existent in this case), which, however, is located outside of asymptotic region. Therefore, to compute $\beta_{CS}(g^2)$ in given approximation, we must preserve at least three first terms in (117). All the next terms are necessary only in the case when smooth 'link-ups' are needed between (117) and $\beta_{CS}(g^2)$ computed with the help (116) in a strong coupling area $g \gg 1$.

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¹¹The asymptotic expansion (115) gives reliable results only for $M \gg 2\beta e$, so for $\beta > \beta_{asynpt}$ it would be legal up to 9-th term.

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